TECHNICAL MEMORANDUMS

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

No. 1017

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Luftfahrtforschung
Vol. 18, No. 10, October 27, 1941
Verlag von R. Oldenbourg, München und Berlin

Washington
June 1942

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SUMMARY

As a contribution to the problem of turbulence on a surface of rotation, the method of small oscillations is applied to the flow past a sphere. It was found that the method developed for two-dimensional flow is applicable without modifications. The frictional layer in the vicinity of the stagnation point of a surface of rotation is less stable against small two-dimensional disturbances than in the stagnation point itself, as proved from an analysis of the velocity distribution made by Homann.

INTRODUCTION

The prediction of the resistance of moving bodies is predicated on the knowledge of the conditions under which the laminar flow in the frictional layer adjacent to the wall becomes turbulent. The analytical treatments by Tollmien (references 1, 2), Schlichtung (references 3, 4, 5, 6), and Görtler (references 7, 8) dealt with the two-dimensional problem. These investigations were based upon the method of small oscillations by ascertaining whether the small oscillations of various frequencies superposed on the basic flow are amplified or damped at given Reynolds numbers. A critical Reynolds number, the so-called "stability limit," can be considered as a first orientating measure of the stability of laminar flow, below which the oscillations of all frequencies for the velocity profile concerned are exactly damped. This stability limit can be indicated for the velocity profiles accompanying pressure gradients and pressure rise in a two-dimensional frictional layer and the data of these

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*"Über die Stabilität der Laminarströmung um eine Kugel." Luftfahrtforschung, vol. 18, no. 10, October 27, 1941, pp. 341-344.

calculations used for an approximate prediction of the transitional inception on aircraft wings.

An analysis of a surface of rotation in a flow along its longitudinal axis, such as is approximately represented by the airplane fuselage the resistance of which plays an increasingly important part in the power absorption of aircraft, would be desirable but for the following difficulty:

The stability study requires the curves of the laminar velocity distributions to be known so accurately that even their curvatures can be reliably indicated; but at the present time there is little prospect of attaining this degree of accuracy for a surface of rotation of random section.

For this reason, the stability theory of the laminar frictional layer is to be explored on the simplest surface of rotation, the sphere, for which sufficiently exact velocity distributions in the frictional layer are likely to be available in the near future.

In the concluding chapter, the only reliable distribution known up to now, Homann's stagnation point profile (reference 9), is subjected to a stability analysis.

II. GENERAL DIFFERENTIAL EQUATION OF THE DISTURBANCE

The derivation of the stability theory for the sphere flow proceeds from the general Navier-Stokes equations, expressed in spherical polar coordinates R, ϑ, and Φ (fig. 1) with u, v, and w denoting the velocities in direction of increasing ϑ, R, and Φ, it affords for for the axially symmetrical problem

$$w = 0; \quad \frac{\partial}{\partial \Phi} = 0 \qquad (1)$$

The equation of continuity reads

$$\frac{1}{R \sin \vartheta} \frac{\partial}{\partial \vartheta} (u \sin \vartheta) + \frac{1}{R^2} \frac{\partial}{\partial R}(R^2 v) = 0 \qquad (2)$$

and the equations of motion for u and v have the form

$$\frac{\partial u}{\partial t}+\frac{u}{R}\frac{\partial u}{\partial \vartheta}+v\frac{\partial u}{\partial R}+\frac{uv}{R}=-\frac{1}{\varrho R}\frac{\partial p}{\partial \vartheta}+\nu\left(\frac{\partial^2 u}{\partial R^2}+\frac{2}{R}\frac{\partial u}{\partial R}+\frac{1}{R^2}\frac{\partial^2 u}{\partial \vartheta^2}+\frac{\operatorname{cotg}\vartheta}{R^2}\frac{\partial u}{\partial \vartheta}+\frac{2}{R^2}\frac{\partial v}{\partial \vartheta}-\frac{u}{R^2\sin^2\vartheta}\right), \qquad (3)$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial R}+\frac{u}{R}\frac{\partial v}{\partial \vartheta}-\frac{u^2}{R}=-\frac{1}{\varrho}\frac{\partial p}{\partial R}+\nu\left(\frac{\partial^2 v}{\partial R^2}+\frac{2}{R}\frac{\partial v}{\partial R}+\frac{1}{R^2}\frac{\partial^2 v}{\partial \vartheta^2}+\frac{\operatorname{cotg}\vartheta}{R^2}\frac{\partial v}{\partial \vartheta}-\frac{2v}{R^2}-\frac{2}{R^2}\frac{\partial u}{\partial \vartheta}-\frac{2u\operatorname{cotg}\vartheta}{R^2}\right) \qquad (4)$$

The arc length s starting from stagnation point and the distance n from the sphere surface are now introduced as coordinates

$$s=R_0\vartheta,\qquad n=R-R_0,$$

with R_0 as sphere radius.

Restricted to the flow region wherein $n \ll R_0$, the elimination of pressure from (3) and (4) gives the equation of motion

$$\frac{\partial^2 u}{\partial t\,\partial n}+\frac{\partial u}{\partial n}\frac{\partial u}{\partial s}+u\frac{\partial^2 u}{\partial s\,d n}+\frac{\partial v}{\partial n}\frac{\partial u}{\partial n}+v\frac{\partial^2 u}{\partial n^2}+\frac{v}{R_0}\frac{\partial u}{\partial n}+\frac{u}{R_0}\frac{\partial v}{\partial n}-\frac{\partial^2 v}{\partial t\,d s}-\frac{\partial u}{\partial s}\frac{\partial v}{\partial s}-u\frac{\partial^2 v}{\partial s^2}-\frac{\partial v}{\partial s}\frac{\partial v}{\partial n}-v\frac{\partial^2 v}{\partial s\,\partial n}+\frac{2u}{R_0}\frac{\partial u}{\partial s}=\nu\left[\frac{\partial^3 u}{\partial n^3}+\frac{\partial^3 u}{\partial n\,\partial s^2}+\frac{2}{R_0}\frac{\partial^2 u}{\partial n^2}+\frac{\operatorname{cotg} s/R_0}{R_0}\frac{\partial^2 u}{\partial n\,\partial s}-\frac{1}{R_0^2\sin^2 s/R_0}\frac{\partial u}{\partial n}-\frac{\partial^3 v}{\partial n^2\,\partial s}-\frac{\partial^3 v}{\partial s^3}-\frac{\operatorname{cotg} s/R_0}{R_0}\frac{\partial^2 v}{\partial s^2}+\frac{1}{R_0^2\sin^2 s/R_0}\frac{\partial v}{\partial s}+\frac{2}{R_0^2}\frac{\partial v}{\partial s}+\frac{2}{R_0}\frac{\partial^2 u}{\partial s^2}+\frac{2\operatorname{cotg} s/R_0}{R_0^2}\frac{\partial u}{\partial s}-\frac{2u}{R_0^2\sin^2 s/R_0}\right] \qquad (5)$$

Let U and V designate the tangential and normal components of the axially symmetrical base flow about the sphere having stability to small vibrations is to be analyzed. The superposed interference motion has the stream function

$$\psi(s,n,t)=\varphi(s,n)\,e^{i\alpha(s-ct)}, \qquad (6)$$

α indicating the spatial natural frequency and the real part of $c = c_r + ic_i$ its phase velocity. The interference velocities are given by

$$u^*=-\frac{1}{R_0\sin s/R_0}\frac{\partial\psi}{\partial n}=-\frac{1}{R_0\sin s/R_0}\varphi' e^{i\alpha(s-ct)},$$

$$v^*=\frac{1}{R_0\sin s/R_0}\frac{\partial\psi}{\partial s}=\frac{1}{R_0\sin s/R_0}e^{i\alpha(s-ct)}\left(i\alpha\varphi+\frac{\partial\varphi}{\partial s}\right), \qquad (7)$$

where ' signifies differentiation with respect to n.

The fundamental motion equation of the interference problem then follows the introduction of the flow with velocity components

$$u=U+u^*;\quad v=V+v^*$$

produced by superposition of basic flow and interference motion, in equation (5).

Restricted to the terms linear in φ and with those of subordinate order of magnitude discounted, the general differential equation reads for the present:

$$(U-c)(\varphi''-\alpha^2\varphi)-U''\varphi+\underline{\frac{1}{R_0}(U\varphi'-U'\varphi)}=-\frac{i\nu}{\alpha}[\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi]+\frac{i}{\alpha}[V(\varphi'''-\alpha^2\varphi')-V''\varphi']+\frac{i}{\alpha}\left[(2\alpha^2 c-3\alpha^2 U-U'')\frac{\partial\varphi}{\partial s}+U\frac{\partial\varphi''}{\partial s}\right]+\underline{\frac{i}{\alpha R_0}\operatorname{cotg} s/R_0[(3U-c)\alpha^2\varphi-2(U\varphi''+U'\varphi')]} \qquad (8)$$

by referring the velocities to the velocity of undisturbed flow U_∞ the arc length to the sphere radius R_0 and the wall distance as well as the wave length of the vibration to a reference length of the friction layer, say, the displacement thickness δ^* in a fixed point of the sphere surface equation (8) can also be written the form

$$Re_\infty^*=\frac{U_\infty\delta^*}{\nu}$$

Expressed nondimensionally

$$(U-c)(\varphi''-\alpha^2\varphi)-U''\varphi+\underline{\frac{\delta^*}{R_0}(U\varphi'-U'\varphi)}=-\frac{i}{\alpha Re_\infty^*}[\varphi^{IV}-2\alpha^2\varphi''+\alpha^4\varphi]+\frac{i}{\alpha}[V(\varphi'''-\alpha^2\varphi')-V''\varphi']+\frac{i\delta^*}{\alpha R_0}\left[(2\alpha^2 c-3\alpha^2 U-U'')\frac{\partial\varphi}{\partial s}+U\frac{\partial\varphi''}{\partial s}\right]+\underline{\frac{i\delta^*}{\alpha R_0}\operatorname{cotg} s/R_0[(3U-c)\alpha^2\varphi-2(U\varphi''+U'\varphi')]} \qquad (9)$$

This linear partial differential equation of the fourth order represents the most general motion equation of the disturbed friction layer flow on the sphere. It differs from the corresponding equation of two-dimensional flow by the addition of the underlined terms which have the same order of magnitude as the other terms on the right-hand side of equation (9).

The interference equation applies equally in direct proximity to the stagnation point $s = 0$

because U and c decrease as $\sin s/R_0$, while $\operatorname{cotg} s/R_0 \to \infty$; for $R_0 \to \infty$ it becomes, as expected, the two-dimensional interference equation.

The next problem is to prove that the interference equation for the flow past a sphere has exactly the same particular solutions as the two-dimensional, despite the additive underlined terms.

III. SOLUTIONS

Assuming $\alpha\, Re^*_\infty$ as very large and bearing in mind that

$$\frac{\delta^*}{R_0} = \frac{C}{Re^*_\infty} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; (10)$$

the partial differential equation (9) reduces to Tollmien's frictionless interference equation for two-dimensional flow

$$(U-c)\,(\varphi'' - \alpha^2 \varphi) - U''\varphi = 0, \quad . \; . \; . \; (11)$$

in which "frictionless solutions" φ_1, φ_2 are identical with those of the two-dimensional problem by reason of the same boundary conditions; hence all statements made elsewhere apply here also. In the construction of the solution φ_2 in the critical layer $U = c$ the friction must be taken into account in form of a transitional function, as explained in the following.

The laminar basic velocity in the critical layer is approximate to a parabola

$$U - c = U_0'\,(n - n_c) + \frac{U_0''}{2}(n - n_c)^2 \quad . \; . \; . \; (12)$$

with the new variables

$$\eta = \frac{n - n_c}{\varepsilon};\; \varepsilon = (\alpha\, Re^*_\infty\, U_0')^{-1/3};\; \xi = s, \quad . \; . \; (13)$$

defined as in the two-dimensional problem and

$$\left.\begin{aligned} &\frac{\partial \eta}{\partial n} = \frac{1}{\varepsilon};\; \frac{\partial \eta}{\partial s} = -\frac{1}{\varepsilon}\frac{d n_c}{d s} + \frac{\eta}{3 U_0'}\frac{d U_0'}{d s}, \\ &\frac{\partial}{\partial n} = \frac{1}{\varepsilon}\frac{\partial}{\partial \eta};\; \frac{\partial}{\partial s} = \frac{\partial}{\partial \xi} - \frac{1}{\varepsilon}\frac{d n_c}{d \xi}\frac{\partial}{\partial \eta} + \frac{\eta}{3 U_0'}\frac{d U_0'}{d \xi}\frac{\partial}{\partial \eta} \end{aligned}\right\} \quad (14)$$

the interference equation (9) then reads:

$$\left.\begin{aligned} &\left(U_0'\varepsilon\eta + \frac{U_0''}{2}\varepsilon^2\eta^2\right)\left(\frac{1}{\varepsilon^2}\frac{\partial^2\varphi}{\partial\eta^2} - \alpha^2\varphi\right) - U_0''\varphi + \frac{C}{Re^*_\infty}\left[\frac{1}{\varepsilon}\frac{\partial\varphi}{\partial\eta}\left(c + U_0'\varepsilon\eta + \frac{U_0''}{2}\varepsilon^2\eta^2\right) - \varphi\,(U_0' + U_0''\varepsilon\eta)\right] = \\ &-\frac{i}{\alpha\, Re^*_\infty}\left[\frac{1}{\varepsilon^4}\frac{\partial^4\varphi}{\partial\eta^4} - \frac{2\alpha^2}{\varepsilon^2}\frac{\partial^2\varphi}{\partial\eta^2} + \alpha^4\varphi\right] + \frac{i}{\alpha}\left[V\left(\frac{1}{\varepsilon^3}\frac{\partial^3\varphi}{\partial\eta^3} - \frac{\alpha^2}{\varepsilon}\frac{\partial\varphi}{\partial\eta}\right) - \frac{1}{\varepsilon^3}\frac{\partial^2 V}{\partial\eta^2}\frac{\partial\varphi}{\partial\eta}\right] + \\ &+\frac{iC}{\alpha\, Re^*_\infty}\left\{\left[2\alpha^2 c - 3\alpha^2\left(c + U_0'\varepsilon\eta + \frac{U_0''}{2}\varepsilon^2\eta^2\right) - U_0''\right]\left[\frac{\partial\varphi}{\partial\xi} - \frac{1}{\varepsilon}\frac{d n_c}{d\xi}\frac{\partial\varphi}{\partial\eta} + \frac{\eta}{3U_0}\frac{dU_0'}{d\xi}\frac{\partial\varphi}{\partial\eta}\right]\right. \\ &+\frac{1}{\varepsilon^2}\left[c + U_0'\varepsilon\eta + \frac{U_0''}{2}\varepsilon^2\eta^2\right]\left[\frac{\partial^3\varphi}{\partial\eta^2\partial\xi} + \frac{2}{3U_0'}\frac{dU_0'}{d\xi}\frac{\partial^2\varphi}{\partial\eta^2} - \frac{1}{\varepsilon}\frac{d n_c}{d\xi}\frac{\partial^3\varphi}{\partial\eta^3} + \frac{\eta}{3U_0'}\frac{dU_0'}{d\xi}\frac{\partial^3\varphi}{\partial\eta^3}\right] \\ &\left.+\operatorname{cotg} s/R_0\left[\alpha^2\varphi\left(2c + 3U_0'\varepsilon\eta + \frac{3}{2}U_0''\varepsilon^2\eta^2\right) - \frac{2}{\varepsilon^2}\frac{\partial^2\varphi}{\partial\eta^2}\left(c + U_0'\varepsilon\eta + \frac{U_0''}{2}\varepsilon^2\eta^2\right) - \frac{2}{\varepsilon}\frac{\partial\varphi}{\partial\eta}(U_0' + U_0''\varepsilon\eta)\right]\right\} \end{aligned}\right\} \quad (15)$$

The normal component V of the velocity is obtained from (2), which after omission of the term $\frac{2v}{R_0}$ simplifies to

$$\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n} + \frac{u}{R_0}\operatorname{cotg} s/R_0 = 0 \quad . \; . \; . \; . \; (16)$$

whence

$$\begin{aligned} V = &-\frac{1}{Re^*_\infty}\int_0^n\left(\frac{\partial U}{\partial s} + \frac{U}{R_0}\operatorname{cotg} s/R_0\right) d\tilde{n} \\ = &-\frac{1}{2\,Re^*_\infty}\left\{2\frac{dc}{d\xi}(n_c + \varepsilon\eta) - \frac{dU_0'}{d\xi}(n_c^2 - \varepsilon^2\eta^2)\right. \\ &+\frac{1}{3}\frac{dU_0''}{d\xi}(n_c^3 + \varepsilon^3\eta^3) \\ &-2U_0'\frac{dn_c}{d\xi}(n_c + \varepsilon\eta) + U_0''\frac{dn_c}{d\xi}(n_c^2 - \varepsilon^2\eta^2) \\ &+\frac{1}{R_0}\operatorname{cotg} s/R_0\left[2c\,(n_c + \varepsilon\eta) - U_0'(n_c^2 - \varepsilon^2\eta^2)\right. \\ &\left.\left.+\frac{U_0''}{3}(n_c^3 + \varepsilon^3\eta^3)\right]\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; (17) \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2 V}{\partial\eta^2} = -\frac{1}{Re^*_\infty}&\left[\frac{dU_0'}{d\xi}\varepsilon^2 + \frac{dU_0''}{d\xi}\varepsilon^3\eta - U_0''\frac{dn_c}{d\xi}\varepsilon^2\right. \\ &\left.+\frac{1}{R_0}\operatorname{cotg} s/R_0\,(U_0'\varepsilon^2 + U_0''\varepsilon^3\eta)\right] \quad (18) \end{aligned}$$

Limited to the terms linear in ϵ because of the assumably large Reynolds number Re^*_∞ equation (15) affords, after multiplication by ϵ/U_0'

$$\begin{aligned} &\eta\frac{\partial^2\varphi}{\partial\eta^2} + \frac{U_0''}{2U_0'}\varepsilon\eta^2\frac{\partial^2\varphi}{\partial\eta^2} - \frac{U_0''}{U_0'}\varepsilon\varphi \\ &\quad\underline{+\frac{cC}{Re^*_\infty U_0'}\frac{\partial\varphi}{\partial\eta} + \frac{\varepsilon C}{Re^*_\infty}\left(\eta\frac{\partial\varphi}{\partial\eta} - \varphi\right)} = \\ &\quad-i\frac{\partial^4\varphi}{\partial\eta^4} + i\varepsilon\frac{\partial^3\varphi}{\partial\eta^3}\left[-\frac{dc}{d\xi}\eta_c + \frac{1}{2}\frac{dU_0'}{d\xi}n_c^2\right. \\ &\quad-\frac{1}{6}\frac{dU_0''}{d\xi}n_c^3 + U_0'\frac{dn_c}{d\xi}n_c - \frac{U_0''}{2}\frac{dn_c}{d\xi}\eta_c^2 \\ &\quad\left.-\frac{Ccdn_c}{d\xi} - \frac{1}{R_0}\operatorname{cotg} s/R_0\left(cn_c - \frac{U_0'}{2}n_c^2 + \frac{U_0''}{6}n_c^3\right)\right] \\ &\qquad\qquad . \; . \; . \; . \; . \; . \; (19) \end{aligned}$$

But then

$$\begin{aligned} &\frac{cC}{Re^*_\infty U_0'}\frac{d\varphi}{\partial\eta} + \frac{}{Re_\infty}\left(\eta\frac{\partial\varphi}{\partial\eta} - \varphi\right) \\ &\qquad = Cca\varepsilon^3\frac{\partial\varphi}{\partial\eta} + U_0'\alpha C\varepsilon^4\left(\eta\frac{\partial\varphi}{\partial\eta} - \varphi\right), \quad (20) \end{aligned}$$

so that the underlined terms in (19) are disregarded.

The friction correction for the frictionless flow φ_2 in the critical layer follows from the solution φ_2 in powers of ϵ at

$$\varphi_2 = \varphi + \varepsilon \varphi_{21} + \varepsilon^2 \varphi_{22} + \ldots, \quad \ldots (21)$$

which joins the frictionless solution at some distance from $U = c$.

Posting (21) in (20) we get with $\varphi_{20} = 1$ and multiplying by $-\frac{1}{\epsilon}$:

$$\begin{aligned}\frac{\partial^4 \varphi_{21}}{\partial \eta^4} - i \frac{\partial^2 \varphi_{21}}{\partial \eta^2}\left(\eta + \frac{U_0''}{2 U_0'} \eta^2 \varepsilon\right) = - i \frac{U_0''}{U_0'} (1 + \varepsilon \cdot \varphi_{21}) \\ + \varepsilon \frac{\partial^2 \varphi_{21}}{\partial \eta^2} \Big[- \frac{d c}{d \xi} \eta_c + \frac{1}{2} \frac{d U_0'}{d \xi} n_c^2 \\ - \frac{1}{6} \frac{d U_0''}{d \xi} n_c^3 + U_0' \frac{d \eta_c}{d \xi} n_c - \frac{U_0''}{2} \frac{d n_c}{d \xi} n_c^2 \\ - c C \frac{d n_c}{d \xi} - \frac{1}{R_0} \operatorname{cotg} s/R_0 \left(c n_c - \frac{U_0'}{2} n_c^2 + \frac{U_0''}{6} n_c^3\right)\Big] \\ \ldots\ldots (22)\end{aligned}$$

Again disregarding the terms with ϵ the differential equation for the transitional function φ_{21} is the same as in the two-dimensional problem: namely,

$$\frac{\partial^4 \varphi_{21}}{\partial \eta^4} - i \eta \frac{\partial^2 \varphi_{21}}{\partial \eta^2} = - i \frac{U_0''}{U_0'} \quad \ldots\ldots (23)$$

Finally it can be readily proved that the friction-induced solutions in wall proximity φ_3, φ_4 themselves are as in the two-dimensional problem. For $\epsilon \rightarrow 0$ equation (19) affords the same differential equation

$$i \frac{\partial^4 \varphi_{3,4}}{\partial \eta^4} + \eta \frac{\partial^2 \varphi_{3,4}}{\partial \eta^2} = 0 \quad \ldots\ldots (24)$$

And since the limiting conditions of the differential interference equation for flow past the sphere are also the same as in the two-dimensional problem, it proves that the study of stability of laminar flow past the sphere follows the very same method as in the two-dimensional problem.

This result suggests the suspicion that the method of the two-dimensional problem is not merely applicable to the sphere but generally to any surface of rotation. For rigorous proof of this suspicion we would have to proceed from the equations of motion in general orthogonal coordinates.

IV. THE STABILITY OF LAMINAR FLOW IN THE STAGNATION POINT OF A SURFACE OF ROTATION

The velocity distribution in the stagnation point of the sphere is to be analyzed as regards its stability against small vibrations.

This velocity profile, which has the same form in the stagnation point of any other surface of rotation with other than vanishing curvature radius, was derived by Homann (reference 9) from the hydrodynamic equations of motion expressed for this purpose in cylindrical polar coordinates:

$$\left.\begin{aligned} u \frac{\partial u}{\partial r} + v \frac{\partial u}{\partial z} = - \frac{1}{\varrho} \frac{\partial p}{\partial r} + \nu \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{\partial^2 u}{\partial z^2}\right), \\ u \frac{\partial v}{\partial r} + v \frac{\partial v}{\partial z} = - \frac{1}{\varrho} \frac{\partial p}{\partial z} + \nu \left(\frac{\partial^2 v}{\partial z^2} + \frac{1}{r} \frac{\partial v}{\partial r} + \frac{\partial^2 v}{\partial z^2}\right); \end{aligned}\right\} \quad (25)$$

where u the velocity along the increasing distance r from the stagnation point in the tangential plane at the stagnation point and v the velocity along the increasing distance z from the stagnation point past the body axis.

Outside of the friction layer the potential velocity in the stagnation point is the same as that toward the plate

$$U_a = \beta\ r; \quad V_a = -2\ \beta\ z \qquad (26)$$

with the dimensionless wall distance

$$\zeta = \sqrt{\frac{\beta}{\nu}}\ z \qquad (27)$$

and the formula for the stream function

$$\psi = \sqrt{\beta \nu}\ r^2\ f\ (\zeta) \qquad (28)$$

Homann's differential equation for the stagnation-point profile on the surface of rotation reads:

$$f''' + 2\ f\ f'' - f'^2 + 1 = 0 \quad (29)$$

with $f'(\zeta) = \dfrac{U(\zeta)}{U_a}$ the dimensionless velocity in the friction layer.

It differs from Hiemenz's differential equation for two-dimensional stagnation-point flow merely by the number 2 in the second term; it has the same limiting conditions $f = f' = 0$ for $\zeta = 0$, and $f' = 1$ for $\zeta \to \infty$. The solution is illustrated in figure 2.

For the stability study it further requires an acceptable approximation of the velocity distribution, for which in support of previous calculations (reference 10) an expression of the form

$$\frac{U}{U_a} = 1 - (1 - y)^n \quad (n = 2,3,4 \ldots) \tag{30}$$

is employed, in conjunction with the equation illustrated in figure 2:

$$\frac{U}{U_a} = 1 - (1 - 0.4391\,\zeta)^3 \tag{31}$$

Figure 3 contains the polar diagram for determining the neutral stability curve (fig. 4) which separates stable from unstable interference attitudes in the plane spanned by Reynolds number and interference frequency. The neutral stability curve for the two-dimensional stagnation-point profile according to Hiemenz is included for contrast. It is seen that the stability limit $Re^*_{critical}$ on the three-dimensional stagnation-point profile is about a third of that of the two-dimensional profile. As a consequence, the laminar flow near the stagnation point of a surface of rotation is obviously more unstable in the case of small vibrations than the two-dimensional stagnation-point flow.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.

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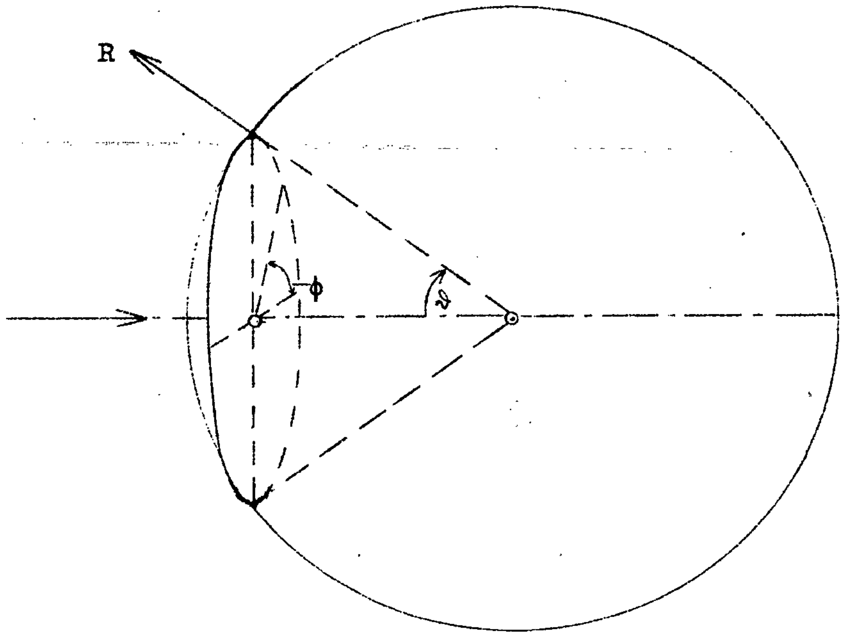

Figure 1.- Definition of spherical polar coordinates.

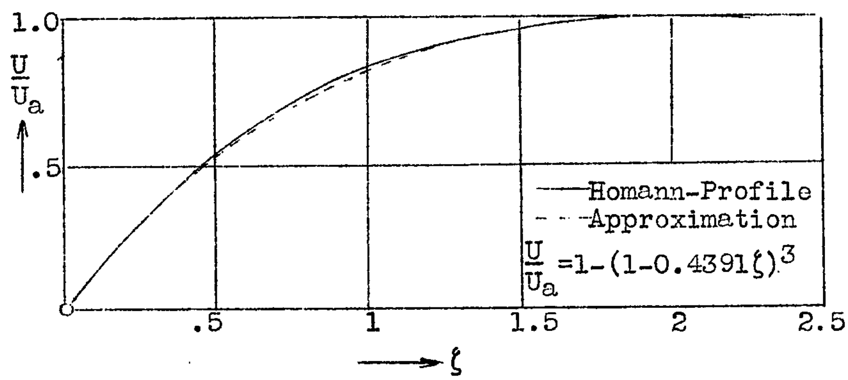

Figure 2.- Velocity distribution in the friction layer at the stagnation point of a surface of rotation.

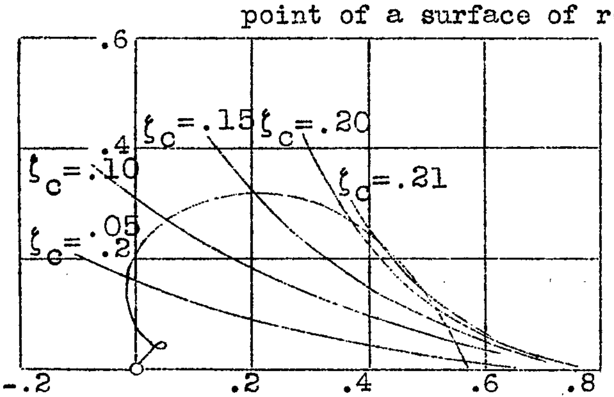

Figure 3.- Polar diagram for determining the neutral stability curve.

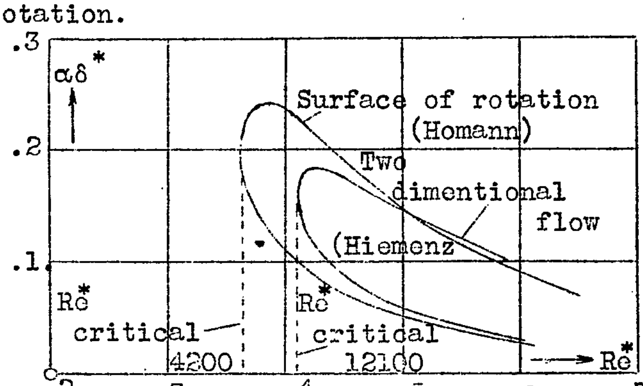

Figure 4.- Neutral stability curve for the stagnation point profile.